Chapter 4. Expansions (Including Substitution)

Exercise 4(A)

Solution 1:

We know that

$(a+b)^2 = a^2 + b^2 + 2ab$

$(2a+b)^2 = 4a^2 + b^2 + 2\times 2a\times b$

$= 4a^2 + b^2 + 4ab$

We know that

$(a+b)^2 = a^2 + b^2 + 2ab$

$(3a+7b)^2 = 9a^2 + 49b^2 + 2\times 3a\times 7b$

$= 9a^2 + 49b^2 + 42ab$

We know that

$(a-b)^2 = a^2 + b^2 - 2ab$

$(3a-4b)^2 = 9a^2 + 16b^2 - 2\times 3a\times 4b$

$= 9a^2 + 16b^2 - 24ab$

We know that

$(a-b)^2 = a^2 + b^2 - 2ab$

$\left(\frac{3a}{2b}-\frac{2b}{3a}\right)^2 = \left(\frac{3a}{2b}\right)^2 + \left(\frac{2b}{3a}\right)^2 - 2\times\frac{3a}{2b}\times\frac{2b}{3a}$

$= \frac{9a^2}{4b^2} + \frac{4b^2}{9a^2} - 2$

Solution 2:

- $(101)^2$

$(101)^2 = (100+1)^2$

We know that

$(a+b)^2 = a^2 + b^2 + 2ab$

$\therefore\ (100+1)^2 = 100^2 + 1^2 + 2\times 100\times 1$

$= 10000 + 1 + 200$

$= 10,201$

- $(502)^2$

$(502)^2 = (500+2)^2$

We know that

$(a+b)^2 = a^2 + b^2 + 2ab$

$\therefore\ (500+2)^2 = 500^2 + 2^2 + 2\times 500\times 2$

$= 250000 + 4 + 2000$

$= 2,52,004$

- $(97)^2$

$(97)^2 = (100-3)^2$

We know that

$(a-b)^2 = a^2 + b^2 - 2ab$

$\therefore (100-3)^2 = 100^2 + 3^2 - 2 \times 100 \times 3$

$= 10000 + 9 - 600$

$= 9,409$

- $(998)^2$

$(998)^2 = (1000-2)^2$

We know that

$(a-b)^2 = a^2 + b^2 - 2ab$

$\therefore (1000-2)^2 = 1000^2 + 2^2 - 2 \times 1000 \times 2$

$= 1000000 + 4 - 4000$

$= 9,96,004$

Solution 3:

(i)

$$\left(\frac{7}{8}x+\frac{4}{5}y\right)^2$$

We know that

$$(a+b)^2 = a^2+b^2+2ab$$

$$\therefore \left(\frac{7}{8}x+\frac{4}{5}y\right)^2 = \left(\frac{7}{8}x\right)^2+\left(\frac{4}{5}y\right)^2+2\times\frac{7}{8}x\times\frac{4}{5}y$$

$$= \frac{49x^2}{64}+\frac{16y^2}{25}+\frac{7xy}{5}$$

(ii)

$$\left(\frac{2x}{7}-\frac{7y}{4}\right)^2$$

We know that

$$(a-b)^2 = a^2+b^2-2ab$$

$$\therefore \left(\frac{2x}{7}-\frac{7y}{4}\right)^2 = \left(\frac{2}{7}x\right)^2+\left(\frac{7}{4}y\right)^2-2\times\frac{2}{7}x\times\frac{7}{4}y$$

$$= \frac{4x^2}{49}+\frac{49y^2}{16}-xy$$

Solution 4:

(i) Consider the given expression:

Let us expand the first term: $\left(\frac{a}{2b}+\frac{2b}{a}\right)^2$

We know that

$$(a+b)^2 = a^2+b^2+2ab$$

$$\therefore \left(\frac{a}{2b}+\frac{2b}{a}\right)^2 = \left(\frac{a}{2b}\right)^2+\left(\frac{2b}{a}\right)^2+2\times\frac{a}{2b}\times\frac{2b}{a}$$

$$=\frac{a^2}{4b^2}+\frac{4b^2}{a^2}+2\ldots(1)$$

Let us expand the second term: $\left(\frac{a}{2b}-\frac{2b}{a}\right)^2$

We know that

$$(a-b)^2 = a^2+b^2-2ab$$

$$\therefore \left(\frac{a}{2b}-\frac{2b}{a}\right)^2 = \left(\frac{a}{2b}\right)^2+\left(\frac{2b}{a}\right)^2-2\times\frac{a}{2b}\times\frac{2b}{a}$$

$$=\frac{a^2}{4b^2}+\frac{4b^2}{a^2}-2\ldots(2)$$

Thus from (1) and (2), the given expression is

$$\left(\frac{a}{2b}+\frac{2b}{a}\right)^2-\left(\frac{a}{2b}-\frac{2b}{a}\right)^2-4=\frac{a^2}{4b^2}+\frac{4b^2}{a^2}+2-\frac{a^2}{4b^2}-\frac{4b^2}{a^2}+2-4$$

$$=0$$

(ii) Consider the given expression:

Let us expand the first term: $(4a+3b)^2$

We know that

$(a+b)^2 = a^2 + b^2 + 2ab$

$\therefore (4a+3b)^2 = (4a)^2 + (3b)^2 + 2 \times 4a \times 3b$

$= 16a^2 + 9b^2 + 24ab \ldots(1)$

Let us expand the second term: $(4a-3b)^2$

We know that

$(a-b)^2 = a^2 + b^2 - 2ab$

$\therefore (4a-3b)^2 = (4a)^2 + (3b)^2 - 2 \times 4a \times 3b$

$= 16a^2 + 9b^2 - 24ab \ldots(2)$

Thus from (1) and (2), the given expression is

$(4a+3b)^2 - (4a-3b)^2 + 48ab$

$= 16a^2 + 9b^2 + 24ab - 16a^2 - 9b^2 + 24ab + 48ab$

$= 96ab$

Solution 5:

We know that

$(a+b)^2 = a^2 + b^2 + 2ab$

and

$(a-b)^2 = a^2 + b^2 - 2ab$

Rewrite the above equation, we have

$(a-b)^2 = a^2 + b^2 + 2ab - 4ab$

$= (a+b)^2 - 4ab \ldots(1)$

Given that a+b = 7; ab=10

Substitute the values of (a+b) and (ab)

in equation (1), we have

$(a-b)^2 = (7)^2 - 4(10)$

$= 49 - 40 = 9$

$\Rightarrow a - b = \pm\sqrt{9}$

$\Rightarrow a - b = \pm 3$

Solution 6:

We know that

$(a-b)^2 = a^2 + b^2 - 2ab$

and

$(a+b)^2 = a^2 + b^2 + 2ab$

Rewrite the above equation, we have

$(a+b)^2 = a^2 + b^2 - 2ab + 4ab$

$= (a-b)^2 + 4ab \ldots(1)$

Given that $a-b = 7$; $ab=18$

Substitute the values of $(a-b)$ and (ab) in equation (1), we have

$(a+b)^2 = (7)^2 + 4(18)$

$= 49 + 72 = 121$

$\Rightarrow a+b = \pm\sqrt{121}$

$\Rightarrow a+b = \pm 11$

Solution 7:

(i)

We know that

$(x+y)^2 = x^2 + y^2 + 2xy$

and

$(x-y)^2 = x^2 + y^2 - 2xy$

Rewrite the above equation, we have

$(x-y)^2 = x^2 + y^2 + 2xy - 4xy$

$= (x+y)^2 - 4xy \ldots(1)$

Given that $x+y = \frac{7}{2}$; $xy=\frac{5}{2}$

Substitute the values of $(x+y)$ and (xy) in equation (1), we have

$(x-y)^2 = \left(\frac{7}{2}\right)^2 - 4\left(\frac{5}{2}\right)$

$= \frac{49}{4} - 10 = \frac{9}{4}$

$\Rightarrow x-y = \pm\sqrt{\frac{9}{4}}$

$\Rightarrow x-y = \pm\frac{3}{2} \ldots(2)$

(ii)

We know that

$x^2 - y^2 = (x+y)(x-y) \ldots.(3)$

From equation (2) we have,

$x - y = \pm\frac{3}{2}$

Thus equation (3) becomes,

$x^2 - y^2 = \left(\frac{7}{2}\right)\left(\pm\frac{3}{2}\right)$ [given $x + y = \frac{7}{2}$]

$\Rightarrow x^2 - y^2 = \pm\frac{21}{4}$

Solution 8:

(i)

We know that

$(a-b)^2 = a^2 + b^2 - 2ab$

and

$(a+b)^2 = a^2 + b^2 + 2ab$

Rewrite the above equation, we have

$(a+b)^2 = a^2 + b^2 - 2ab + 4ab$

$= (a-b)^2 + 4ab....(1)$

Given that $a-b = 0.9$; $ab=0.36$

Substitute the values of $(a-b)$ and (ab) in equation (1), we have

$(a+b)^2 = (0.9)^2 + 4(0.36)$

$= 0.81 + 1.44 = 2.25$

$\Rightarrow a+b = \pm\sqrt{2.25}$

$\Rightarrow a+b = \pm 1.5....(2)$

(ii)

We know that

$a^2 - b^2 = (a+b)(a-b)....(3)$

From equation (2) we have,

$a+b = \pm 1.5$

Thus equation (3) becomes,

$a^2 - b^2 = (\pm 1.5)(0.9)$ [given $a-b = 0.9$]

$\Rightarrow a^2 - b^2 = \pm 1.35$

Solution 9:

(i)

We know that

$(a-b)^2 = a^2 + b^2 - 2ab$

Rewrite the above identity as,

$a^2 + b^2 = (a-b)^2 + 2ab \ldots(1)$

Similarly, we know that,

$(a+b)^2 = a^2 + b^2 + 2ab$

Rewrite the above identity as,

$a^2 + b^2 = (a+b)^2 - 2ab \ldots(2)$

Adding the equations (1) and (2), we have

$2(a^2 + b^2) = (a-b)^2 + 2ab + (a+b)^2 - 2ab$

$\Rightarrow 2(a^2 + b^2) = (a-b)^2 + (a+b)^2$

$\Rightarrow (a^2 + b^2) = \frac{1}{2}\left[(a-b)^2 + (a+b)^2\right] \ldots(3)$

Given that $a+b = 6$; $a-b=4$

Substitute the values of $(a+b)$ and $(a-b)$

in equation (3), we have

$(a^2 + b^2) = \frac{1}{2}\left[(4)^2 + (6)^2\right]$

$= \frac{1}{2}[16 + 36]$

$= \frac{52}{2}$

$\Rightarrow a^2 + b^2 = 26 \ldots(4)$

(ii)

From equation (4), we have

$a^2 + b^2 = 26$

Consider the identity

$(a-b)^2 = a^2 + b^2 - 2ab \ldots(5)$

Subsitute the value $a-b = 4$ and $a^2 + b^2 = 26$

in the above equation, we have

$(4)^2 = 26 - 2ab$

$\Rightarrow 2ab = 26 - 16$

$\Rightarrow 2ab = 10$

$\Rightarrow ab = 5$

Solution 10:

(i)

We know that

$(a+b)^2 = a^2 + b^2 + 2ab$

and

$(a-b)^2 = a^2 + b^2 - 2ab$

Thus,

$$\left(a+\frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} + 2 \times a \times \frac{1}{a}$$

$$= a^2 + \frac{1}{a^2} + 2.....(1)$$

Given that $a + \frac{1}{a} = 6$; Substitute in equation (1), we have

$$(6)^2 = a^2 + \frac{1}{a^2} + 2$$

$$\Rightarrow a^2 + \frac{1}{a^2} = 36 - 2$$

$$\Rightarrow a^2 + \frac{1}{a^2} = 34....(2)$$

Similarly, consider

$$\left(a-\frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} - 2 \times a \times \frac{1}{a}$$

$$= a^2 + \frac{1}{a^2} - 2$$

$$= 34 - 2 \text{ [from (2)]}$$

$$\Rightarrow \left(a-\frac{1}{a}\right)^2 = 32$$

$$\Rightarrow a - \frac{1}{a} = \pm\sqrt{32}$$

$$\Rightarrow a - \frac{1}{a} = \pm 4\sqrt{2} \quad .....(3)$$

(ii)

We need to find $a^2 - \frac{1}{a^2}$:

We know that, $a^2 - \frac{1}{a^2} = \left(a-\frac{1}{a}\right)\left(a+\frac{1}{a}\right)$

$a - \frac{1}{a} = \pm 4\sqrt{2}$; $a + \frac{1}{a} = 6$

Thus,

$$a^2 - \frac{1}{a^2} = \left(\pm 4\sqrt{2}\right)(6)$$

$$\Rightarrow a^2 - \frac{1}{a^2} = \pm 24\sqrt{2}$$

Solution 11:

(i)

We know that

$(a+b)^2 = a^2 + b^2 + 2ab$

and

$(a-b)^2 = a^2 + b^2 - 2ab$

Thus,

$$\left(a - \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} - 2 \times a \times \frac{1}{a}$$

$$= a^2 + \frac{1}{a^2} - 2.....(1)$$

Given that $a - \frac{1}{a} = 8$; Substitute in equation (1), we have

$$(8)^2 = a^2 + \frac{1}{a^2} - 2$$

$$\Rightarrow a^2 + \frac{1}{a^2} = 64 + 2$$

$$\Rightarrow a^2 + \frac{1}{a^2} = 66....(2)$$

Similarly, consider

$$\left(a + \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} + 2 \times a \times \frac{1}{a}$$

$$= a^2 + \frac{1}{a^2} + 2$$

$$= 66 + 2 \text{ [from (2)]}$$

$$\Rightarrow \left(a + \frac{1}{a}\right)^2 = 68$$

$$\Rightarrow a + \frac{1}{a} = \pm 2\sqrt{17}$$

$$\Rightarrow a + \frac{1}{a} = \pm 2\sqrt{17} \quad .....(3)$$

(ii)

We need to find $a^2 - \frac{1}{a^2}$:

We know that, $a^2 - \frac{1}{a^2} = \left(a - \frac{1}{a}\right)\left(a + \frac{1}{a}\right)$

$a - \frac{1}{a} = 8; a + \frac{1}{a} = \pm 2\sqrt{17}$

Thus,

$a^2 - \frac{1}{a^2} = \left(\pm 2\sqrt{17}\right)(8)$

$\Rightarrow a^2 - \frac{1}{a^2} = \pm 16\sqrt{17}$

Solution 12:

(i)

Consider the given equation

$a^2 - 3a + 1 = 0$

Rewrite the given equation, we have

$a^2 + 1 = 3a$

$\Rightarrow \frac{a^2 + 1}{a} = 3$

$\Rightarrow \frac{a^2}{a} + \frac{1}{a} = 3$

$\Rightarrow a + \frac{1}{a} = 3 \ldots .(1)$

(ii)

We need to find $a^2 + \frac{1}{a^2}$:

We know the identity, $(a+b)^2 = a^2 + b^2 + 2ab$

$\therefore \left(a + \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} + 2....(2)$

From equation (1), we have,

$a + \frac{1}{a} = 3$

Thus equation (2), becomes,

$(3)^2 = a^2 + \frac{1}{a^2} + 2$

$\Rightarrow 9 = a^2 + \frac{1}{a^2} + 2$

$\Rightarrow a^2 + \frac{1}{a^2} = 7$

Solution 13:

(i)

Consider the given equation

$a^2 - 5a - 1 = 0$

Rewrite the given equation, we have

$a^2 - 1 = 5a$

$\Rightarrow \frac{a^2 - 1}{a} = 5$

$\Rightarrow \frac{a^2}{a} - \frac{1}{a} = 5$

$\Rightarrow a - \frac{1}{a} = 5....(1)$

(ii)

We need to find $a+\frac{1}{a}$:

We know the identity, $(a-b)^2 = a^2+b^2-2ab$

$\therefore \left(a-\frac{1}{a}\right)^2 = a^2+\frac{1}{a^2}-2$

$\Rightarrow (5)^2 = a^2+\frac{1}{a^2}-2$ [from (1)]

$\Rightarrow 25 = a^2+\frac{1}{a^2}-2$

$\Rightarrow a^2+\frac{1}{a^2} = 27 \ldots\ldots(2)$

Now consider the identity $(a+b)^2 = a^2+b^2+2ab$

$\therefore \left(a+\frac{1}{a}\right)^2 = a^2+\frac{1}{a^2}+2$

$\Rightarrow \left(a+\frac{1}{a}\right)^2 = 27+2$ [from (2)]

$\Rightarrow \left(a+\frac{1}{a}\right)^2 = 29$

$\Rightarrow a+\frac{1}{a} = \pm\sqrt{29} \ldots\ldots(3)$

Solution 14:

Given that $(3x+4y)=16$ and $xy=4$

We need to find $9x^2+16y^2$.

We know that

$(a+b)^2 = a^2+b^2+2ab$

Consider the square of $3x+4y$:

$\therefore (3x+4y)^2 = (3x)^2+(4y)^2+2\times 3x\times 4y$

$\Rightarrow (3x+4y)^2 = 9x^2+16y^2+24xy.....(1)$

Substitute the values of $(3x+4y)$ and xy

in the above equation (1), we have

$(3x+4y)^2 = 9x^2+16y^2+24xy$

$\Rightarrow (16)^2 = 9x^2+16y^2+24(4)$

$\Rightarrow 256 = 9x^2+16y^2+96$

$\Rightarrow 9x^2+16y^2 = 160$

Solution 15:

Given x is 2 more than y, so $x = y + 2$

Sum of squares of x and y is 34, so $x^2 + y^2 = 34$.

Replace $x = y + 2$ in the above equation and solve for y.

We get $(y + 2)^2 + y^2 = 34$

$2y^2 + 4y - 30 = 0$

$y^2 + 2y - 15 = 0$

$(y + 5)(y - 3) = 0$

So $y = -5$ or 3

For $y = -5$, $x = -3$

For $y = 3$, $x = 5$

Product of x and y is 15 in both cases.

Solution 16:

Let the two positive numbers be a and b.

Given difference between them is 5 and sum of squares is 73.

So $a - b = 5$, $a^2 + b^2 = 73$

Squaring on both sides gives

$(a - b)^2 = 5^2$

$a^2 + b^2 - 2ab = 25$

but $a^2 + b^2 = 73$

so $2ab = 73 - 25 = 48$

$ab = 24$

So, the product of numbers is 24.

Exercise 4(B)

Solution 1:

(i)

$$(a-b)^3 = a^3 - 3ab(a-b) - b^3$$

$$(3a-2b)^3 = (3a)^3 - 3\times 3a\times 2b(3a-2b) - (2b)^3$$

$$= 27a^3 - 18ab(3a-2b) - 8b^3$$

$$= 27a^3 - 54a^2b + 36ab^2 - 8b^3$$

(ii)

$$(a+b)^3 = a^3 + 3ab(a+b) + b^3$$

$$(5a+3b)^3 = (5a)^3 + 3\times 5a\times 3b(5a+3b) + (3b)^3$$

$$= 125a^3 + 45ab(5a+3b) + 27b^3$$

$$= 125a^3 + 225a^2b + 135ab^2 + 27b^3$$

(iii)

$$(a+b)^3 = a^3 + 3ab(a+b) + b^3$$

$$\left(2a+\frac{1}{2a}\right)^3 = (2a)^3 + 3\times 2a\times\frac{1}{2a}\times\left(2a+\frac{1}{2a}\right)+\left(\frac{1}{2a}\right)^3$$

$$= 8a^3 + 3\left(2a+\frac{1}{2a}\right)+\frac{1}{8a^3}$$

$$\left(2a+\frac{1}{2a}\right)^3 = 8a^3 + 6a + \frac{3}{2a} + \frac{1}{8a^3}$$

(iv)

$$(a-b)^3 = a^3 - 3ab(a-b) - b^3$$

$$\left(3a-\frac{1}{a}\right)^3 = (3a)^3 - 3\times 3a\times\frac{1}{a}\left(3a-\frac{1}{a}\right)-\left(\frac{1}{a}\right)^3$$

$$= 27a^3 - 9\left(3a-\frac{1}{a}\right)-\frac{1}{a^3}$$

$$= 27a^3 - 27a + \frac{9}{a} - \frac{1}{a^3}$$

Solution 2:

(i)

$$a^2 + \frac{1}{a^2} = 47$$

$$\left(a+\frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} + 2$$

$$\Rightarrow \left(a+\frac{1}{a}\right)^2 = 47 + 2$$

$$\Rightarrow \left(a+\frac{1}{a}\right)^2 = 49$$

$$\Rightarrow a + \frac{1}{a} = \pm\sqrt{49}$$

$$\Rightarrow a + \frac{1}{a} = \pm 7 \ldots .(1)$$

(ii)

$$\left(a+\frac{1}{a}\right)^3 = a^3+\frac{1}{a^3}+3\left(a+\frac{1}{a}\right)$$

$$\Rightarrow a^3+\frac{1}{a^3} = \left(a+\frac{1}{a}\right)^3 - 3\left(a+\frac{1}{a}\right)$$

$$\Rightarrow a^3+\frac{1}{a^3} = (\pm 7)^3 - 3(\pm 7) \quad [\text{from } (1)]$$

$$\Rightarrow a^3+\frac{1}{a^3} = \pm 322$$

Solution 3:

(i)

$$a^2+\frac{1}{a^2} = 18$$

$$\left(a-\frac{1}{a}\right)^2 = a^2+\frac{1}{a^2}-2$$

$$\Rightarrow \left(a-\frac{1}{a}\right)^2 = 18-2$$

$$\Rightarrow \left(a-\frac{1}{a}\right)^2 = 16$$

$$\Rightarrow a-\frac{1}{a} = \pm\sqrt{16}$$

$$\Rightarrow a-\frac{1}{a} = \pm 4 \ldots.(1)$$

(ii)

$$\left(a-\frac{1}{a}\right)^3 = a^3-\frac{1}{a^3}-3\left(a-\frac{1}{a}\right)$$

$$\Rightarrow a^3-\frac{1}{a^3} = \left(a-\frac{1}{a}\right)^3 + 3\left(a-\frac{1}{a}\right)$$

$$\Rightarrow a^3-\frac{1}{a^3} = (\pm 4)^3 + 3(\pm 4) \quad [\text{from } (1)]$$

$$\Rightarrow a^3-\frac{1}{a^3} = \pm 76$$

Solution 4:

Given that $a+\frac{1}{a}=p....(1)$

$$\left(a+\frac{1}{a}\right)^3=a^3+\frac{1}{a^3}+3\left(a+\frac{1}{a}\right)$$

$$\Rightarrow a^3+\frac{1}{a^3}=\left(a+\frac{1}{a}\right)^3-3\left(a+\frac{1}{a}\right)$$

$$\Rightarrow a^3+\frac{1}{a^3}=(p)^3-3(p)\ \text{[from (1)]}$$

$$\Rightarrow a^3+\frac{1}{a^3}=p\left(p^2-3\right)$$

Solution 5:

Given that a+2b=5;

We need to find a^3+8b^3+30ab :

Now consider the cube of a+2b:

$$(a+2b)^3=a^3+(2b)^3+3\times a\times 2b\times(a+2b)$$

$$=a^3+8b^3+6ab\times(a+2b)$$

$$5^3=a^3+8b^3+6ab\times(5)\ [\because a+2b=5]$$

$$125=a^3+8b^3+30ab$$

Thus the value of a^3+8b^3+30ab is 125.

Solution 6:

Given that $\left(a+\frac{1}{a}\right)^2 = 3$

$\Rightarrow a+\frac{1}{a} = \pm\sqrt{3}....(1)$

We need to find $a^3+\frac{1}{a^3}$:

Consider the identity,

$\left(a+\frac{1}{a}\right)^3 = a^3+\frac{1}{a^3}+3\left(a+\frac{1}{a}\right)$

$\Rightarrow a^3+\frac{1}{a^3} = \left(\pm\sqrt{3}\right)^3 - 3\left(\pm\sqrt{3}\right)$ [from (1)]

$\Rightarrow a^3+\frac{1}{a^3} = \pm3\sqrt{3} - 3\left(\pm\sqrt{3}\right)$

$\Rightarrow a^3+\frac{1}{a^3} = 0$

Solution 7:

Given that a+2b+c=0;

$\Rightarrow$ a+2b=– c....(1)

Now consider the expansion of $(a+2b)^3$:

$$(a+2b)^3 = (-c)^3$$

$a^3+(2b)^3+3\times a\times 2b\times(a+2b) = -c^3$

$\Rightarrow \quad a^3+8b^3+3\times a\times 2b\times(-c) = -c^3 \quad$ [from (1)]

$\Rightarrow \quad a^3+8b^3-6abc = -c^3$

$\Rightarrow \quad a^3+8b^3+c^3 = 6abc$

Hence proved.

Solution 8:

Property is if a + b + c = 0 then $a^3 + b^3 + c^3 = 3abc$

(i) a = 13, b = -8 and c = -5

$13^3 + (-8)^3 + (-5)^3 = 3(13)(-8)(-5) = 1560$

(ii) a = 7, b = 3, c = -10

$7^3 + 3^3 + (-10)^3 = 3(7)(3)(-10) = -630$

(iii)a = 9, b = -5, c = -4

$9^3 - 5^3 - 4^3 = 9^3 + (-5)^3 + (-4)^3 = 3(9)(-5)(-4) = 540$

(iv) a = 38, b = -26, c = -12

$38^3 + (-26)^3 + (-12)^3 = 3(38)(-26)(-12) = 35568$

Solution 9:

(i)

$$a - \frac{1}{a} = 3$$

$$\left(a - \frac{1}{a}\right)^2 = 9$$

$$a^2 + \frac{1}{a^2} = 9 + 2 = 11$$

(ii)

$$a - \frac{1}{a} = 3$$

$$\left(a - \frac{1}{a}\right)^3 = 27$$

$$a^3 + \frac{1}{a^3} - 3\left(a - \frac{1}{a}\right) = 27$$

$$a^3 + \frac{1}{a^3} = 27 + 9 = 36$$

Solution 10:

(i)

$$\left(a-\frac{1}{a}\right)^2 = a^2+\frac{1}{a^2}-2$$

$$\Rightarrow a^2+\frac{1}{a^2} = \left(a-\frac{1}{a}\right)^2+2$$

$$\Rightarrow a^2+\frac{1}{a^2} = (4)^2+2 \quad [\because a-\frac{1}{a}=4]$$

$$\Rightarrow a^2+\frac{1}{a^2} = 18.....(1)$$

(ii)

We know that

$$a^4+\frac{1}{a^4} = \left(a^2+\frac{1}{a^2}\right)^2-2$$

$$= (18)^2-2 \quad [\text{from } (1)]$$

$$= 324-2$$

$$\Rightarrow a^4+\frac{1}{a^4} = 322$$

(iii)

$$\left(a-\frac{1}{a}\right)^3 = a^3-\frac{1}{a^3}-3\left(a-\frac{1}{a}\right)$$

$$\Rightarrow a^3-\frac{1}{a^3} = \left(a-\frac{1}{a}\right)^3+3\left(a-\frac{1}{a}\right)$$

$$\Rightarrow a^3-\frac{1}{a^3} = (4)^3+3(4) \quad [\because a-\frac{1}{a}=4]$$

$$\Rightarrow a^3-\frac{1}{a^3} = 64+12$$

$$\Rightarrow a^3-\frac{1}{a^3} = 76$$

Solution 11:

$$\left(x+\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}+2$$

$$\Rightarrow x^2+\frac{1}{x^2} = \left(x+\frac{1}{x}\right)^2 - 2$$

$$\Rightarrow x^2+\frac{1}{x^2} = (2)^2 - 2 \quad \left[\because x-\frac{1}{x}=2\right]$$

$$\Rightarrow x^2+\frac{1}{x^2} = 2.....(1)$$

$$\left(x+\frac{1}{x}\right)^3 = x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right)$$

$$\Rightarrow x^3+\frac{1}{x^3} = \left(x+\frac{1}{x}\right)^3 - 3\left(x+\frac{1}{x}\right)$$

$$\Rightarrow x^3+\frac{1}{x^3} = (2)^3 - 3(2) \quad \left[\because x+\frac{1}{x}=2\right]$$

$$\Rightarrow x^3+\frac{1}{x^3} = 8-6$$

$$\Rightarrow x^3+\frac{1}{x^3} = 2....(2)$$

We know that

$$x^4+\frac{1}{x^4} = \left(x^2+\frac{1}{x^2}\right)^2 - 2$$

$$= (2)^2 - 2 \quad [\text{from } (1)]$$

$$= 4-2$$

$$\Rightarrow x^4+\frac{1}{x^4} = 2.....(3)$$

Thus from equations (1), (2) and (3), we have

$$x^2+\frac{1}{x^2} = x^3+\frac{1}{x^3} = x^4+\frac{1}{x^4}$$

Solution 12:

Given that $2x - 3y = 10, xy = 16$

$\therefore (2x - 3y)^3 = (10)^3$

Þ $8x^3 - 27y^3 - 3(2x)(3y)(2x - 3y) = 1000$ Þ $8x^3 - 27y^3 - 18xy(2x - 3y) = 1000$

Þ $8x^3 - 27y^3 - 18(16)(10) = 1000$

Þ $8x^3 - 27y^3 - 2880 = 1000$

Þ $8x^3 - 27y^3 = 1000 + 2880$

Þ $8x^3 - 27y^3 = 3880$

Solution 13:

(i)

$(3x + 5y + 2z)(3x - 5y + 2z)$

$= \{(3x + 2z) + (5y)\}\{(3x + 2z) - (5y)\}$

$= (3x + 2z)^2 - (5y)^2$

$\{\text{since } (a + b)(a - b) = a^2 - b^2\}$

$= 9x^2 + 4z^2 + 2 \times 3x \times 2z - 25y^2$

$= 9x^2 + 4z^2 + 12xz - 25y^2$

$= 9x^2 + 4z^2 - 25y^2 + 12xz$

(ii)

$(3x - 5y - 2z)(3x - 5y + 2z)$

$= \{(3x - 5y) - (2z)\}\{(3x - 5y) + (2z)\}$

$= (3x - 5y)^2 - (2z)^2 \{\text{since} (a + b)(a - b) = a^2 - b^2\}$

$= 9x^2 + 25y^2 - 2 \times 3x \times 5y - 4z^2$

$= 9x^2 + 25y^2 - 30xy - 4z^2$

$= 9x^2 + 25y^2 - 4z^2 - 30xy$

Solution 14:

Given sum of two numbers is 9 and their product is 20.

Let the numbers be a and b.

$a + b = 9$

$ab = 20$

Squaring on both sides gives

$(a+b)^2 = 9^2$

$a^2 + b^2 + 2ab = 81$

$a^2 + b^2 + 40 = 81$

So sum of squares is 81 - 40 = 41

Cubing on both sides gives

$(a + b)^3 = 9^3$

$a^3 + b^3 + 3ab(a + b) = 729$

$a^3 + b^3 + 60(9) = 729$

$a^3 + b^3 = 729 - 540 = 189$

So the sum of cubes is 189.

Solution 15:

Cubing on both sides gives

$(x - y)^3 = 5^3$

$x^3 - y^3 - 3xy(x - y) = 125$

$x^3 - y^3 - 72(5) = 125$

$x^3 - y^3 = 125 + 360 = 485$

So, difference of their cubes is 485.

Cubing both sides, we get

$(x + y)^3 = 11^3$

$x^3 + y^3 + 3xy(x + y) = 1331$

$x^3 + y^3 = 1331 - 72(11) = 1331 - 792 = 539$

So, sum of their cubes is 539.

Exercise 4(C)

Solution 1:

(i) $(x+8)(x+10) = x^2 + (8+10)x + 8 \times 10$
$= x^2 + 18x + 80$

(ii) $(x+8)(x-10) = x^2 + (8-10)x + 8 \times (-10)$
$= x^2 - 2x - 80$

(iii) $(x-8)(x+10) = x^2 - (8-10)x - 8 \times 10$
$= x^2 + 2x - 80$

(iv) $(x-8)(x-10) = x^2 - (8+10)x + 8 \times 10$
$= x^2 - 18x + 80$

Solution 2:

(i) $\left(2x-\frac{1}{x}\right)\left(3x+\frac{2}{x}\right)=(2x)(3x)-\left(\frac{1}{x}\right)(3x)+\left(\frac{2}{x}\right)(2x)-\left(\frac{1}{x}\right)\left(\frac{2}{x}\right)$

$=6x^2-(3-2)-\frac{2}{x^2}$

$=6x^2-(-1)-\frac{2}{x^2}$

$=6x^2+1-\frac{2}{x^2}$

(ii) $\left(3a+\frac{2}{b}\right)\left(2a-\frac{3}{b}\right)=(3a)(2a)+\left(\frac{2}{b}\right)(2a)-\left(\frac{3}{b}\right)(3a)-\left(\frac{2}{b}\right)\left(\frac{3}{b}\right)$

$=6a^2+\left(\frac{4}{b}-\frac{9}{b}\right)a-\frac{6}{b^2}$

$=6a^2+\left(-\frac{5}{b}\right)a-\frac{6}{b^2}$

$=6a^2-\frac{5a}{b}-\frac{6}{b^2}$

Solution 3:

(i) $(x+y-z)^2=x^2+y^2+z^2+2(x)(y)-2(y)(z)-2(z)(x)$

$=x^2+y^2+z^2+2xy-2yz-2zx$

(ii) $(x-2y+2)^2=x^2+(2y)^2+(2)^2-2(x)(2y)-2(2y)(2)+2(2)(x)$

$=x^2+4y^2+4-4xy-8y+4x$

(iii) $(5a-3b+c)^2=(5a)^2+(3b)^2+(c)^2-2(5a)(3b)-2(3b)(c)+2(c)(5a)$

$=25a^2+9b^2+c^2-30ab-6bc+10ca$

(iv) $(5x-3y-2)=(5x)^2+(3y)^2+(2)^2-2(5x)(3y)+2(3y)(2)-2(2)(5x)$

$=25x^2+9y^2+4-30xy+12y-20x$

(v) $\left(x-\frac{1}{x}+5\right)^2=(x)^2+\left(\frac{1}{x}\right)^2+(5)^2-2(x)\left(\frac{1}{x}\right)-2\left(\frac{1}{x}\right)(5)+2(5)(x)$

$=x^2+\frac{1}{x^2}+25-2-\frac{10}{x}+10x$

$=x^2+\frac{1}{x^2}+23-\frac{10}{x}+10x$

Solution 4:

We know that

$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)....(1)$

Given that, $a^2+b^2+c^2 = 50$ and $a+b+c=12$.

We need to find $ab+bc+ca$:

Substitute the values of $(a^2+b^2+c^2)$ and $(a+b+c)$ in the identity (1), we have

$$(12)^2 = 50+2(ab+bc+ca)$$
$$\Rightarrow 144 = 50+2(ab+bc+ca)$$
$$\Rightarrow 94 = 2(ab+bc+ca)$$
$$\Rightarrow ab+bc+ca = \frac{94}{2}$$
$$\Rightarrow ab+bc+ca = 47$$

Solution 5:

We know that

$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)....(1)$

Given that, $a^2+b^2+c^2 = 35$ and $ab+bc+ca=23$.

We need to find $a+b+c$:

Substitute the values of $(a^2+b^2+c^2)$ and $(ab+bc+ca)$ in the identity (1), we have

$$(a+b+c)^2 = 35+2(23)$$
$$\Rightarrow (a+b+c)^2 = 81$$
$$\Rightarrow a+b+c = \pm\sqrt{81}$$
$$\Rightarrow a+b+c = \pm 9$$

Solution 6:

We know that

$(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)....(1)$

Given that, $a+b+c = p$ and $ab + bc + ca = q$.

We need to find $a^2 + b^2 + c^2$:

Substitute the values of $(ab + bc + ca)$ and $(a+b+c)$ in the identity (1), we have

$$(p)^2 = a^2 + b^2 + c^2 + 2(q)$$
$$\Rightarrow p^2 = a^2 + b^2 + c^2 + 2q$$
$$\Rightarrow a^2 + b^2 + c^2 = p^2 - 2q$$

Solution 7:

$a^2 + b^2 + c^2 = 50$ and $ab + bc + ca = 47$

Since $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\therefore (a + b + c)^2 = 50 + 2(47)$

$\Rightarrow (a + b + c)^2 = 50 + 94 = 144$

$\Rightarrow a + b + c = \sqrt{144} = \pm 12$

$\therefore a + b + c = \pm 12$

Solution 8:

$x + y - z = 4$ and $x^2 + y^2 + z^2 = 30$

Since $(x + y - z)^2 = x^2 + y^2 + z^2 + 2(xy - yz - zx)$, we have

$(4)^2 = 30 + 2(xy - yz - zx)$

$\Rightarrow 16 = 30 + 2(xy - yz - zx)$

$\Rightarrow 2(xy - yz - zx) = -14$

$\Rightarrow xy - yz - zx = \frac{-14}{2} = -7$

$\therefore xy - yz - zx = -7$

Exercise 4(D)

Solution 1:

Given that $x^3 + 4y^3 + 9z^3 = 18xyz$ and $x + 2y + 3z = 0$

\ $x + 2y = -3z$, $2y + 3z = -x$ and $3z + x = -2y$

Now

$$\frac{(x+2y)^2}{xy} + \frac{(2y+3z)^2}{yz} + \frac{(3z+x)^2}{zx} = \frac{(-3z)^2}{xy} + \frac{(-x)^2}{yz} + \frac{(-2y)^2}{zx}$$
$$= \frac{9z^2}{xy} + \frac{x^2}{yz} + \frac{4y^2}{zx}$$
$$= \frac{x^3 + 4y^3 + 9z^3}{xyz}$$

Given that $x^3 + 4y^3 + 9z^3 = 18xyz$

$$\therefore \frac{(x+2y)^2}{xy} + \frac{(2y+3z)^2}{yz} + \frac{(3z+x)^2}{zx} = \frac{18xyz}{xyz} = 18$$

Solution 2:

(i)

Given that $a + \frac{1}{a} = m$;

Now consider the expansion of $\left(a + \frac{1}{a}\right)^2$:

$$\left(a + \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} + 2$$

$$\Rightarrow \quad m^2 = a^2 + \frac{1}{a^2} + 2$$

$$\Rightarrow a^2 + \frac{1}{a^2} = m^2 - 2$$

Now consider the expansion of $\left(a - \frac{1}{a}\right)^2$:

$$\left(a - \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} - 2$$

$$\Rightarrow \left(a - \frac{1}{a}\right)^2 = m^2 - 2 - 2$$

$$\Rightarrow \left(a - \frac{1}{a}\right)^2 = m^2 - 4$$

$$\Rightarrow \left(a - \frac{1}{a}\right) = \pm\sqrt{m^2 - 4} \ldots (1)$$

(ii)

$$a^2 - \frac{1}{a^2} = \left(a + \frac{1}{a}\right)\left(a - \frac{1}{a}\right) \quad [\text{since } a^2 - b^2 = (a+b)(a-b)]$$

$$= m\left(\pm\sqrt{m^2 - 4}\right)$$

$$= \pm m\sqrt{m^2 - 4}$$

Solution 3:

$(2x^2 - 8)(x - 4)^2$

$= (2x^2 - 8)(x^2 - 8x + 16)$

$= 4x^4 - 16x^3 + 32x^2 - 8x^2 + 64x - 128$

$= 4x^4 - 16x^3 + 24x^2 + 64x - 128$

Hence,

coefficient of $x^3 = -16$

coefficient of $x^2 = 24$

constant term $= -128$

Solution 4:

Given that

$$x^2+\frac{1}{9x^2}=\frac{25}{36}$$

$$\Rightarrow x^2+\frac{1}{(3x)^2}=\frac{25}{36}\ldots.(1)$$

Now consider the expansion of $\left(x+\frac{1}{3x}\right)^2$:

$$\left(x+\frac{1}{3x}\right)^2=x^2+\frac{1}{(3x)^2}+2\times x\times\frac{1}{3x}$$

$$\Rightarrow \qquad =x^2+\frac{1}{(3x)^2}+\frac{2}{3}$$

$$\Rightarrow \qquad =\frac{25}{36}+\frac{2}{3} \quad [\text{from }(1)]$$

$$\Rightarrow \qquad =\frac{25+24}{36}$$

$$\Rightarrow \qquad =\frac{49}{36}$$

$$\Rightarrow x+\frac{1}{3x}=\pm\sqrt{\frac{49}{36}}$$

$$\Rightarrow x+\frac{1}{3x}=\pm\frac{7}{6}\ldots.(2)$$

Solution 5:

(i)

$$2(x^2+1)=5x$$

$$\left(x^2+1\right)=\frac{5}{2}x$$

Dividing by x, we have

$$\frac{\left(x^2+1\right)}{x}=\frac{5}{2}$$

$$\Rightarrow\left(x+\frac{1}{x}\right)=\frac{5}{2}\ldots.(1)$$

Now consider the expansion of $\left(x+\frac{1}{x}\right)^2$:

$$\left(x+\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}+2$$

$$\Rightarrow \left(\frac{5}{2}\right)^2 = x^2+\frac{1}{x^2}+2 \text{ [from (1)]}$$

$$\Rightarrow \left(\frac{5}{2}\right)^2 - 2 = x^2+\frac{1}{x^2}$$

$$\Rightarrow \frac{25}{4} - 2 = x^2+\frac{1}{x^2}$$

$$\Rightarrow x^2+\frac{1}{x^2} = \frac{25-8}{4}$$

$$\Rightarrow x^2+\frac{1}{x^2} = \frac{17}{4} \ldots.(2)$$

Now consider the expansion of $\left(x-\frac{1}{x}\right)^2$:

$$\left(x-\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}-2$$

$$\Rightarrow \left(x-\frac{1}{x}\right)^2 = \frac{17}{4}-2 \text{ [from (2)]}$$

$$\Rightarrow \left(x-\frac{1}{x}\right)^2 = \frac{17-8}{4}$$

$$\Rightarrow \left(x-\frac{1}{x}\right)^2 = \frac{9}{4}$$

$$\Rightarrow \left(x-\frac{1}{x}\right) = \pm\frac{3}{2} \ldots..(3)$$

(ii)

We know that,

$$\left(x^3-\frac{1}{x^3}\right) = \left(x-\frac{1}{x}\right)^3 + 3\left(x-\frac{1}{x}\right)$$

$$\therefore \left(x^3-\frac{1}{x^3}\right) = \left(\pm\frac{3}{2}\right)^3 + 3\left(\pm\frac{3}{2}\right) \text{ [from (3)]}$$

$$= \pm\frac{27}{8}+\frac{9}{2}$$

$$\Rightarrow \left(x^3-\frac{1}{x^3}\right) = \pm\frac{27+36}{8}$$

$$\Rightarrow \left(x^3-\frac{1}{x^3}\right) = \pm\frac{63}{8}$$

Solution 6:

$a^2 + b^2 = 34$, $ab = 12$

$(a + b)^2 = a^2 + b^2 + 2ab$

$= 34 + 2 \times 12 = 34 + 24 = 58$

$(a - b)^2 = a^2 + b^2 - 2ab$

$= 34 - 2 \times 12 = 34 - 24 = 10$

(i) $3(a + b)^2 + 5(a - b)^2$

$= 3 \times 58 + 5 \times 10 = 174 + 50$

$= 224$

(ii) $7(a - b)^2 - 2(a + b)^2$

$= 7 \times 10 - 2 \times 58 = 70 - 116 = -46$

Solution 7:

Given $3x - \frac{4}{x} = 4$;

We need to find $27x^3 - \frac{64}{x^3}$

Let us now consider the expansion of $\left(3x - \frac{4}{x}\right)^3$:

$$\left(3x - \frac{4}{x}\right)^3 = 27x^3 - \frac{64}{x^3} - 3 \times 3x \times \frac{4}{x}\left(3x - \frac{4}{x}\right)$$

$$\Rightarrow (4)^3 = 27x^3 - \frac{64}{x^3} - 144 \quad \left[\text{Given}: 3x - \frac{4}{x} = 4\right]$$

$$\Rightarrow 64 + 144 = 27x^3 - \frac{64}{x^3}$$

$$\Rightarrow 27x^3 - \frac{64}{x^3} = 208$$

Solution 8:

Given that $x^2 + \frac{1}{x^2} = 7$

We need to find the value of $7x^3 + 8x - \frac{7}{x^3} - \frac{8}{x}$

Consider the given equation:

$$x^2 + \frac{1}{x^2} - 2 = 7 - 2 \text{ [subtract 2 from both the sides]}$$

$$\Rightarrow \left(x - \frac{1}{x}\right)^2 = 5$$

$$\Rightarrow \left(x - \frac{1}{x}\right) = \pm\sqrt{5} \ldots\ldots(1)$$

$$\therefore 7x^3 + 8x - \frac{7}{x^3} - \frac{8}{x} = 7x^3 - \frac{7}{x^3} + 8x - \frac{8}{x}$$

$$= 7\left(x^3 - \frac{1}{x^3}\right) + 8\left(x - \frac{1}{x}\right) \ldots\ldots(2)$$

Now consider the expansion of $\left(x - \frac{1}{x}\right)^3$:

$$\left(x - \frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right)$$

$$\Rightarrow x^3 - \frac{1}{x^3} = \left(x - \frac{1}{x}\right)^3 + 3\left(x - \frac{1}{x}\right)$$

$$\Rightarrow x^3 - \frac{1}{x^3} = \left(\sqrt{5}\right)^3 + 3\left(\sqrt{5}\right) \ldots\ldots(3)$$

Now substitute the value of $x^3 - \frac{1}{x^3}$ in equation (2), we have

$$7x^3 + 8x - \frac{7}{x^3} - \frac{8}{x} = 7\left(x^3 - \frac{1}{x^3}\right) + 8\left(x - \frac{1}{x}\right)$$

$$\Rightarrow 7x^3 + 8x - \frac{7}{x^3} - \frac{8}{x} = 7\left[\left(\sqrt{5}\right)^3 + 3\left(\sqrt{5}\right)\right] + 8\left[\sqrt{5}\right] \text{ [from (3)]}$$

$$\Rightarrow 7x^3 + 8x - \frac{7}{x^3} - \frac{8}{x} = 7\left[5\left(\sqrt{5}\right) + 3\left(\sqrt{5}\right)\right] + 8\left[\sqrt{5}\right]$$

$$\Rightarrow 7x^3 + 8x - \frac{7}{x^3} - \frac{8}{x} = 64\sqrt{5}$$

Solution 9:

Given $x = \dfrac{1}{x-5}$;

By cross multiplication,

=> x (x - 5) = 1 => x^2 - 5x = 1 => x^2 - 1 = 5x
Dividing both sides by x,

$$\frac{x^2 - 1}{x} = 5$$

$$\Rightarrow \left(x - \frac{1}{x}\right) = 5 \ldots.(1)$$

Now consider the expansion of $\left(x - \frac{1}{x}\right)^2$:

$$\left(x - \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} - 2$$

$$\Rightarrow (5)^2 = x^2 + \frac{1}{x^2} - 2$$

$$\Rightarrow x^2 + \frac{1}{x^2} = 25 + 2 = 27 \ldots.(2)$$

Let us consider the expansion of $\left(x + \frac{1}{x}\right)^2$:

$$\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = 27 + 2 \quad [\text{from } (2)]$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = 29$$

$$\Rightarrow \left(x + \frac{1}{x}\right) = \pm\sqrt{29} \ldots..(3)$$

We know that

$$x^2 - \frac{1}{x^2} = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right)$$

$$= \left(\pm\sqrt{29}\right)(5) \quad [\text{from equations } (1) \text{ and } (3)]$$

$$\Rightarrow x^2 - \frac{1}{x^2} = \pm 5\sqrt{29}$$

Solution 10:

Given $x = \frac{1}{5-x}$;

By cross multiplication,

=> x (5 - x) = 1 => x^2 - 5x =-1 => x^2 + 1 = 5x
Dividing both sides by x,

$$\frac{x^2+1}{x} = 5$$
$$\Rightarrow \left(x + \frac{1}{x}\right) = 5....(1)$$

We know that

$$x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right)$$
$$= (5)^3 - 3(5) \quad \text{[from equation (1)]}$$
$$\Rightarrow x^3 + \frac{1}{x^3} = 125 - 15 = 110$$

Solution 11:

Given that 3a + 5b + 4c = 0

3a + 5b = -4c

Cubing both sides,

$(3a + 5b)^3 = (-4c)^3$

=>$(3a)^3 + (5b)^3$ + 3 x 3a x 5b (3a + 5b) = $-64c^3$

=>$27a^3 + 125b^3$ + 45ab x (-4c) = $-64c^3$

=>$27a^3 + 125b^3 - 180abc = -64c^3$

=>$27a^3 + 125b^3 + 64c^3 = 180abc$

Hence proved.

Solution 12:

Let a, b be the two numbers

$\therefore$ a + b = 7 and $a^3 + b^3 = 133$

$(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

$\Rightarrow (7)^3 = 133 + 3ab(7)$

$\Rightarrow 343 = 133 + 21ab \Rightarrow 21ab = 343 - 133 = 210$

$\Rightarrow 21ab = 210 \Rightarrow ab = 2l$

Now $a^2 + b^2 = (a + b)^2 - 2ab$

$= 7^2 - 2 \times 10 = 49 - 20 = 29$

Solution 13:

(i) $4x^2 + ax + 9 = (2x + 3)^2$

Comparing coefficients of x terms, we get

ax = 12x

so, a = 12

(ii) $4x^2 + ax + 9 = (2x - 3)^2$

Comparing coefficients of x terms, we get

ax = -12x

so, a = -12

(iii) $9x^2 + (7a - 5)x + 25 = (3x + 5)^2$

Comparing coefficients of x terms, we get

(7a - 5)x = 30x

7a - 5 = 30

7a = 35

a = 5

Solution 14:

Given

$$\frac{x^2+1}{x}=\frac{10}{3}$$

$$x+\frac{1}{x}=\frac{10}{3}$$

Squaring on both sides. we get

$$x^2+\frac{1}{x^2}+2=\frac{100}{9}$$

$$x^2+\frac{1}{x^2}=\frac{100-18}{9}=\frac{82}{9}$$

$$x-\frac{1}{x}=\sqrt{\left(x+\frac{1}{x}\right)^2-4}=\sqrt{\frac{100}{9}-4}=\sqrt{\frac{64}{9}}=\frac{8}{3}$$

$$\therefore\ x-\frac{1}{x}=\frac{8}{3}$$

Cubing both sides, we get

$$\left(x-\frac{1}{x}\right)^3=\frac{512}{27}$$

$$x^3-\frac{1}{x^3}-3\left(x-\frac{1}{x}\right)=\frac{512}{27}$$

$$x^3-\frac{1}{x^3}=\frac{512}{27}+8=\frac{512+216}{27}=\frac{728}{27}$$

Solution 15:

Given difference between two positive numbers is 4 and difference between their cubes is 316.

Let the positive numbers be a and b

$a - b = 4$

$a^3 - b^3 = 316$

Cubing both sides,

$(a - b)^3 = 64$

$a^3 - b^3 - 3ab(a - b) = 64$

Given $a^3 - b^3 = 316$

So $316 - 64 = 3ab(4)$

$252 = 12ab$

So $ab = 21$; product of numbers is 21

Squaring both sides, we get

$(a - b)^2 = 16$

$a^2 + b^2 - 2ab = 16$

$a^2 + b^2 = 16 + 42 = 58$

Sum of their squares is 58.

Exercise 4(E)

Solution 1:

Using identity:

$(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$

(i) $(x + 6)(x + 4)(x - 2)$

$= x^3 + (6 + 4 - 2)x^2 + [6 \times 4 + 4 \times (-2) + (-2) \times 6]x + 6 \times 4 \times (-2)$

$= x^3 + 8x^2 + (24 - 8 - 12)x - 48$

$= x^3 + 8x^2 + 4x - 48$

(ii) $(x - 6)(x - 4)(x + 2)$

$= x^3 + (-6 - 4 + 2)x^2 + [-6 \times (-4) + (-4) \times 2 + 2 \times (-6)]x + (-6) \times (-4) \times 2$

$= x^3 - 8x^2 + (24 - 8 - 12)x + 48$

$= x^3 - 8x^2 + 4x + 48$

(iii) $(x - 6)(x - 4)(x - 2)$

$= x^3 + (-6 - 4 - 2)x^2 + [-6 \times (-4) + (-4) \times (-2) + (-2) \times (-6)]x + (-6) \times (-4) \times (-2)$

$= x^3 - 12x^2 + (24 + 8 + 12)x - 48$

$= x^3 - 12x^2 + 44x - 48$

(iv) $(x + 6)(x - 4)(x - 2)$

$= x^3 + (6 - 4 - 2)x^2 + [6 \times (-4) + (-4) \times (-2) + (-2) \times 6]x + 6 \times (-4) \times (-2)$

$= x^3 - 0x^2 + (-24 + 8 - 12)x + 48$

$= x^3 - 28x + 48$

Solution 2:

(i) $(2x+3y)(4x^2-6xy+9y^2) = (2x+3y)\left[(2x)^2-(2x)(3y)+(3y)^2\right]$

$= (2x)^3+(3y)^3$

$= 8x^3+27y^3$

(ii) $\left(3x-\frac{5}{x}\right)\left(9x^2+15+\frac{25}{x^2}\right) = \left(3x-\frac{5}{x}\right)\left((3x)^2+(3x)\left(\frac{5}{x}\right)+\left(\frac{5}{x}\right)^2\right)$

$= (3x)^3-\left(\frac{5}{x}\right)^3$

$= 27x^3-\frac{125}{x^3}$

(iii) $\left(\frac{a}{3}-3b\right)\left(\frac{a^2}{9}+ab+9b^2\right) = \left(\frac{a}{3}-3b\right)\left(\left(\frac{a}{3}\right)^2+\left(\frac{a}{3}\right)(3b)+(3b)^2\right)$

$= \left(\frac{a}{3}\right)^3-(3b)^3$

$= \frac{a^3}{27}-27b^3$

Solution 3:

Using identity: $(a \pm b)^3 = a^3 \pm b^3 \pm 3ab(a \pm b)$

(i) $(104)^3 = (100+4)^3$

$= (100)^3 + (4)^3 + 3 \times 100 \times 4(100+4)$

$= 1000000 + 64 + 1200 \times 104$

$= 1000000 + 64 + 124800$

$= 1124864$

(ii) $(97)^3 = (100-3)^3$

$= (100)^3 - (3)^3 - 3 \times 100 \times 3(100-3)$

$= 1000000 - 27 - 900 \times 97$

$= 1000000 - 27 - 87300$

$= 912673$

Solution 4:

$$\frac{\left(x^2-y^2\right)^3+\left(y^2-z^2\right)^3+\left(z^2-x^2\right)^3}{(x-y)^3+(y-z)^3+(z-x)^3}$$

If $a+b+c=0$, then $a^3+b^3+c^3=3abc$

Now, $x^2-y^2+y^2-z^2+z^2-x^2=0$

$$\Rightarrow \left(x^2-y^2\right)^3+\left(y^2-z^2\right)^3+\left(z^2-x^2\right)^3=3\left(x^2-y^2\right)\left(y^2-z^2\right)\left(z^2-x^2\right) \quad ....(1)$$

And, $x-y+y-z+z-x=0$

$$\Rightarrow (x-y)^3+(y-z)^3+(z-x)^3=3(x-y)(y-z)(z-x) \quad ....(2)$$

Now,

$$\frac{\left(x^2-y^2\right)^3+\left(y^2-z^2\right)^3+\left(z^2-x^2\right)^3}{(x-y)^3+(y-z)^3+(z-x)^3}$$

$$=\frac{3\left(x^2-y^2\right)\left(y^2-z^2\right)\left(z^2-x^2\right)}{3(x-y)(y-z)(z-x)} \quad ....\text{[From (1) and (2)]}$$

$$=\frac{(x-y)(x+y)(y-z)(y+z)(z-x)(z+x)}{(x-y)(y-z)(z-x)}$$

$$=(x+y)(y+z)(z+x)$$

Solution 5:

(i) $\dfrac{0.8\times0.8\times0.8+0.5\times0.5\times0.5}{0.8\times0.8-0.8\times0.5+0.5\times0.5}$

Let $0.8 = a$ and $0.5 = b$

Then, the given expression becomes

$$\frac{a\times a\times a+b\times b\times b}{a\times a-a\times b+b\times b}$$

$$=\frac{a^3+b^3}{a^2-ab+b^2}$$

$$=\frac{(a+b)(a^2-ab+b^2)}{a^2-ab+b^2}$$

$$=a+b$$

$$=0.8+0.5$$

$$=1.3$$

(ii) $\dfrac{1.2\times1.2+1.2\times0.3+0.3\times0.3}{1.2\times1.2\times1.2-0.3\times0.3\times0.3}$

Let $1.2 = a$ and $0.3 = b$

Then, the given expression becomes

$$\frac{a\times a+a+b+b\times b}{a\times a\times a-b\times b\times b}$$

$$=\frac{a^2+ab+b^2}{a^3-b^3}$$

$$=\frac{a^2+ab+b^2}{(a-b)(a^2+ab+b^2)}$$

$$=\frac{1}{a-b}$$

$$=\frac{1}{1.2-0.3}$$

$$=\frac{1}{0.9}$$

$$=\frac{10}{9}$$

$$=1\frac{1}{9}$$

Solution 6:

$a^3 - 8b^3 + 27c^3 = a^3 + (-2b)^3 + (3c)^3$

Since $a - 2b + 3c = 0$, we have

$a^3 - 8b^3 + 27c^3 = a^3 + (-2b)^3 + (3c)^3$

$= 3(a)(-2b)(3c)$

$= -18abc$

Solution 7:

$x + 5y = 10$

$\Rightarrow (x + 5y)^3 = 10^3$

$\Rightarrow x^3 + (5y)^3 + 3(x)(5y)(x + 5y) = 1000$

$\Rightarrow x^3 + (5y)^3 + 3(x)(5y)(10) = 1000$

$= x^3 + (5y)^3 + 150xy = 1000$

$= x^3 + (5y)^3 + 150xy - 1000 = 0$

Solution 8:

$x = 3 + 2\sqrt{2}$

(i) $\frac{1}{x} = \frac{1}{3 + 2\sqrt{2}}$

$= \frac{1}{3 + 2\sqrt{2}} \times \frac{3 - 2\sqrt{2}}{3 - 2\sqrt{2}}$

$= \frac{3 - 2\sqrt{2}}{(3)^2 - (2\sqrt{2})^2}$

$= \frac{3 - 2\sqrt{2}}{9 - 8}$

$\therefore \frac{1}{x} = 3 - 2\sqrt{2} \quad ....(1)$

(ii) $x - \frac{1}{x} = (3 + 2\sqrt{2}) - (3 - 2\sqrt{2}) \quad ....[\text{From } (1)]$

$= 3 + 2\sqrt{2} - 3 + 2\sqrt{2}$

$\therefore x - \frac{1}{x} = 4\sqrt{2} \quad ....(2)$

(iii) $\left(x - \frac{1}{x}\right)^3 = (4\sqrt{2})^3 \quad ....[\text{From } (2)]$

$= 64 \times 2\sqrt{2}$

$= 128\sqrt{2}$

(iv) $x^3 - \frac{1}{x^3} = \left(x - \frac{1}{x}\right)^3 + 3\left(x - \frac{1}{x}\right)$

$= 128\sqrt{2} + 3(4\sqrt{2})$

$= 128\sqrt{2} + 12\sqrt{2}$

Solution 9:

$a + b = 11$ and $a^2 + b^2 = 65$

Now, $(a+b)^2 = a^2 + b^2 + 2ab$

$\Rightarrow (11)^2 = 65 + 2ab$

$\Rightarrow 121 = 65 + 2ab$

$\Rightarrow 2ab = 56$

$\Rightarrow ab = 28$

$$\begin{aligned} a^3 + b^3 &= (a+b)(a^2 - ab + b^2) \\ &= (11)(65 - 28) \\ &= 11 \times 37 \\ &= 407 \end{aligned}$$